The Closure Property of Data Types

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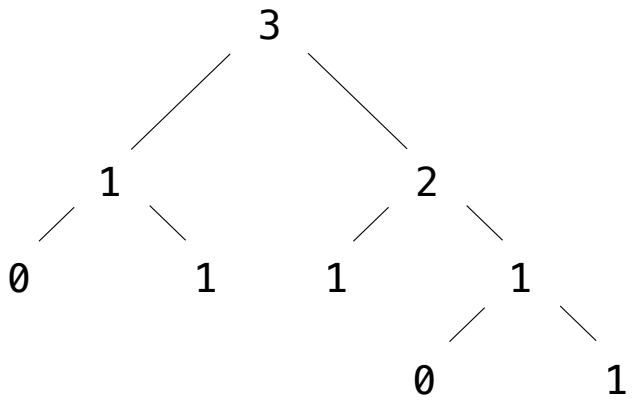
- Hierarchical structures are made up of parts, which themselves are made up of parts, and so on

Lists can contain lists as elements (in addition to anything else)

Trees

Tree Abstraction



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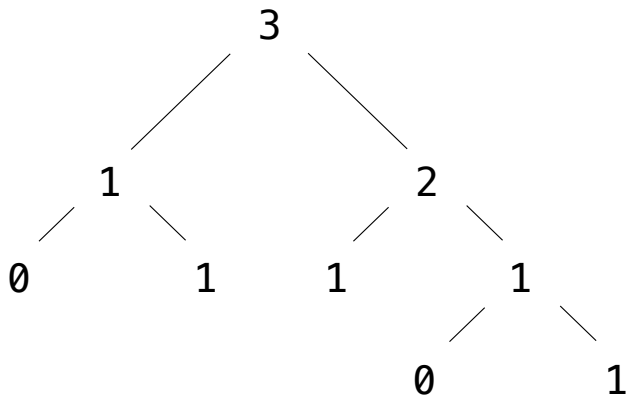

Recursive description (wooden trees):

Relative description (family trees):

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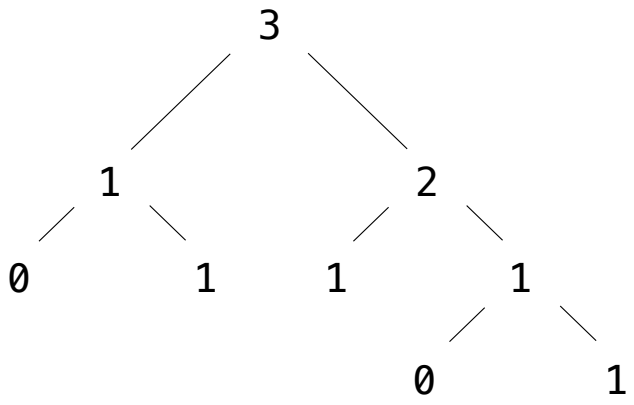

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A **tree** has a **root** and a list of **branches**

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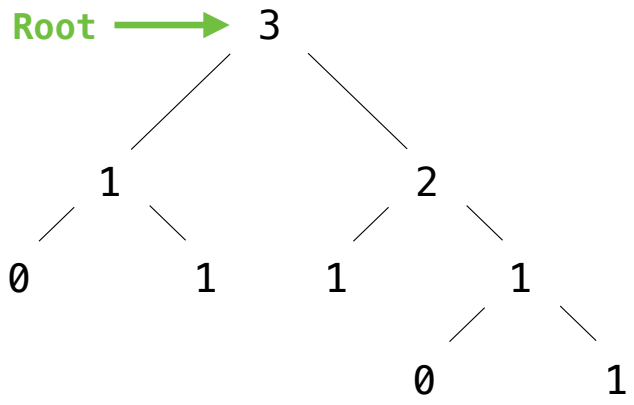

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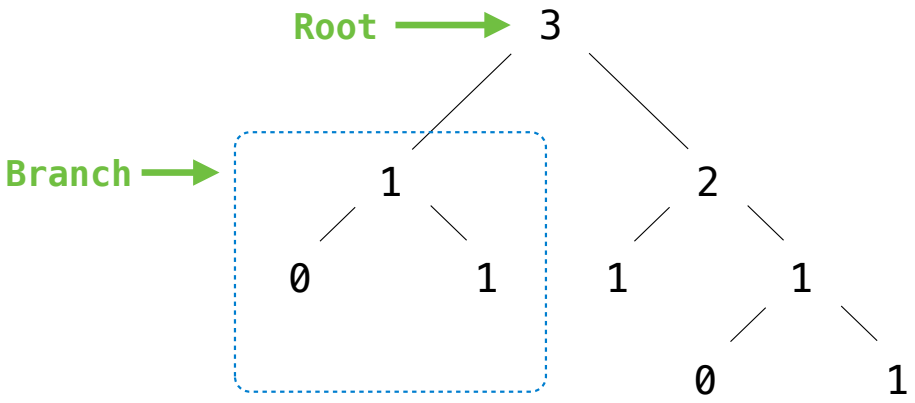

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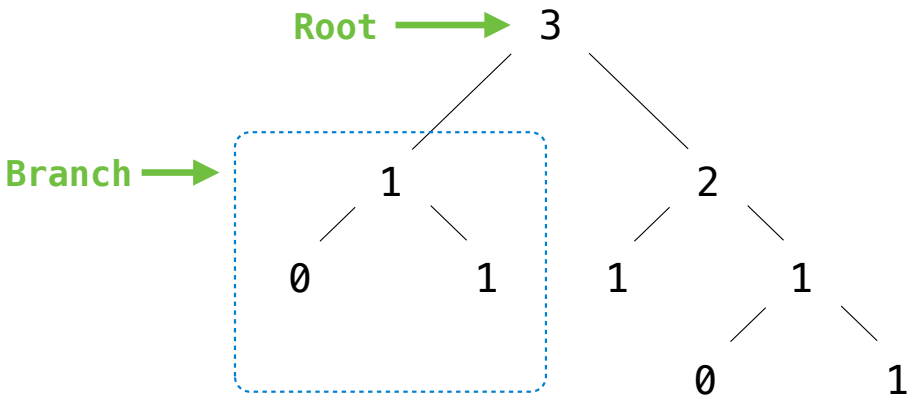

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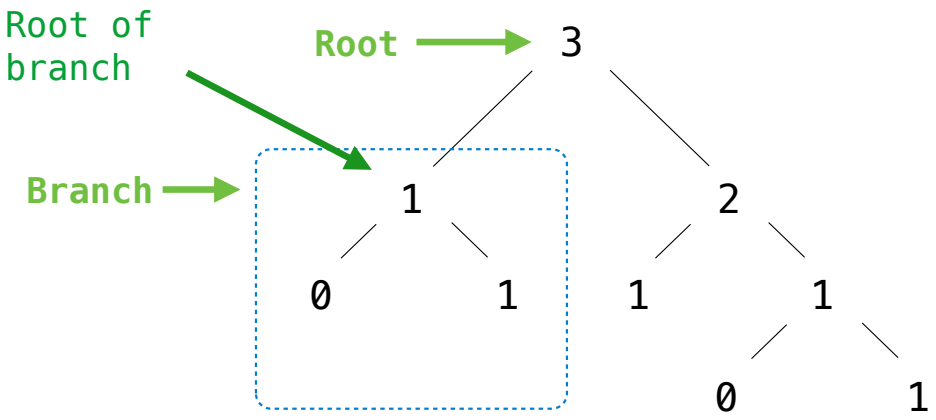

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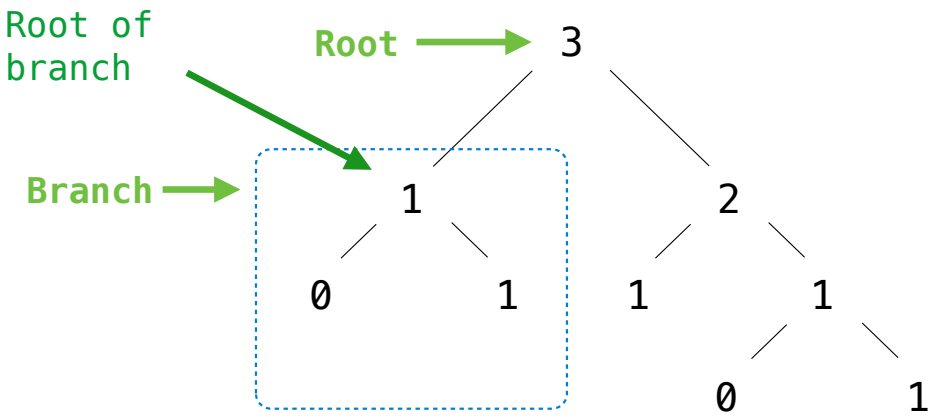

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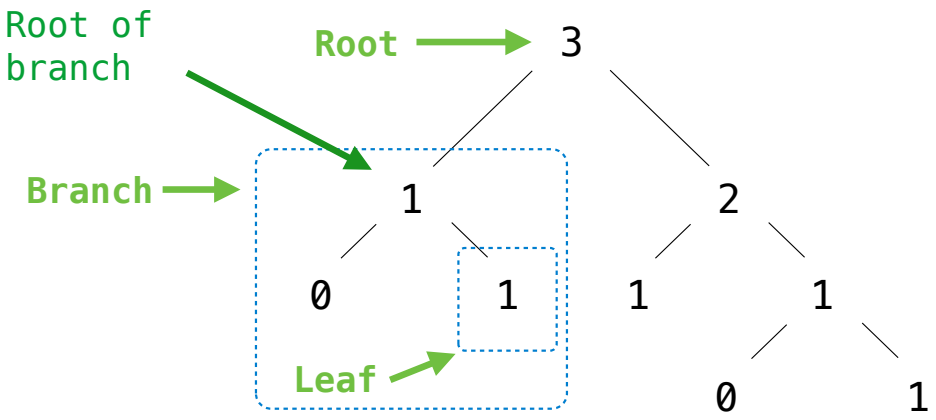

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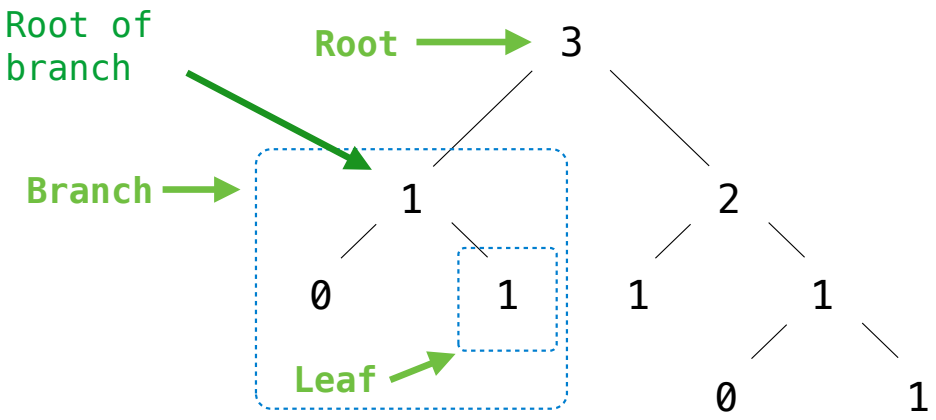

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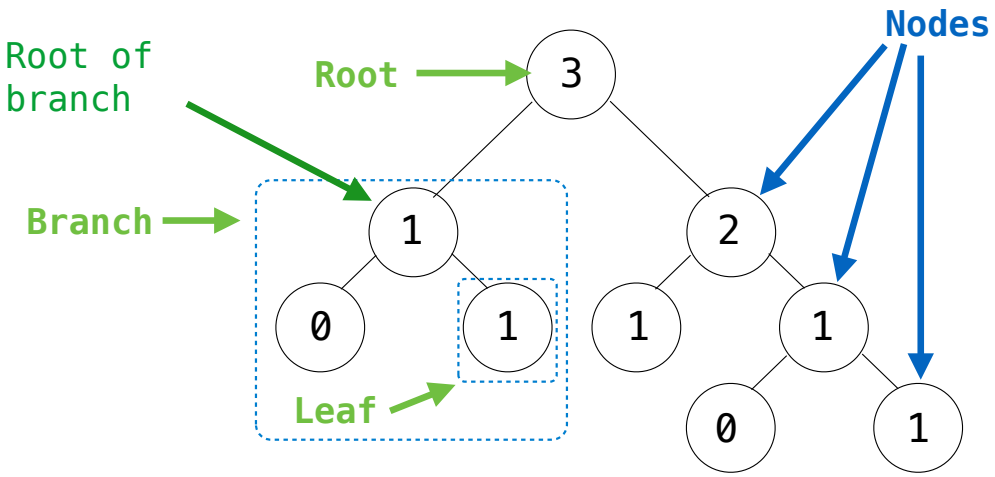

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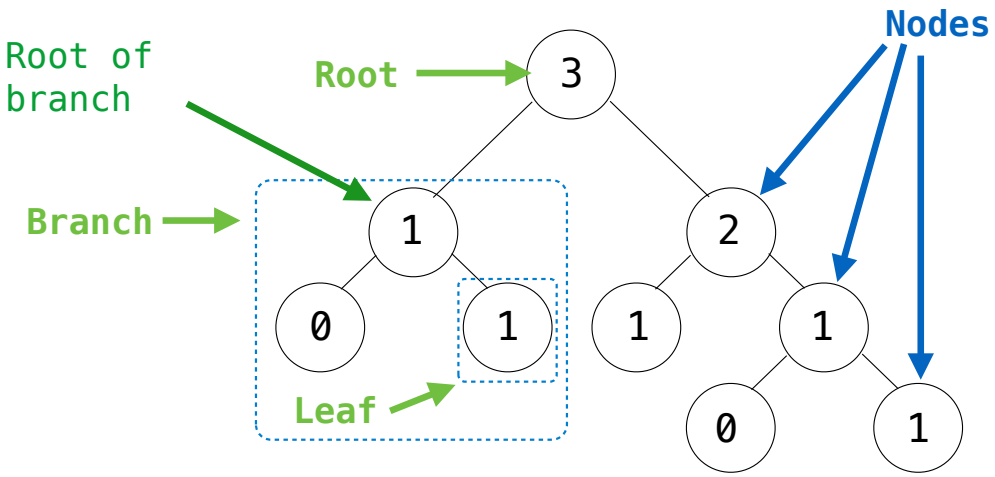

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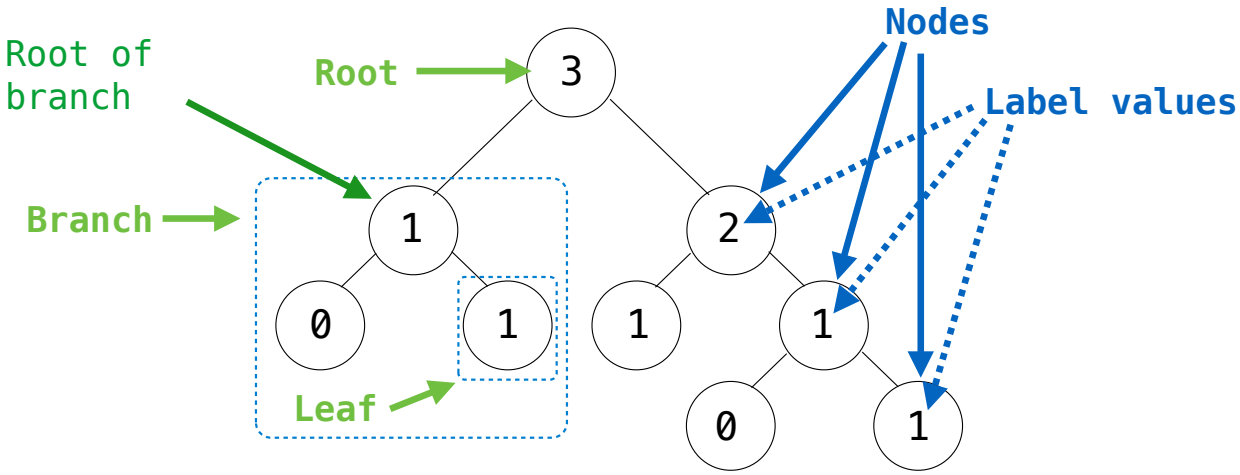

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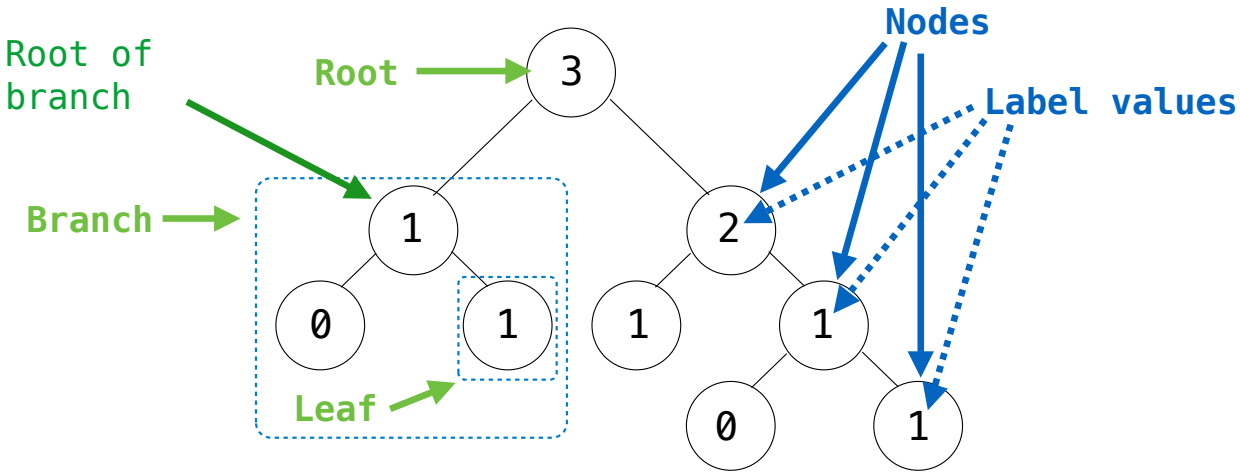

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One node can be the **parent/child** of another

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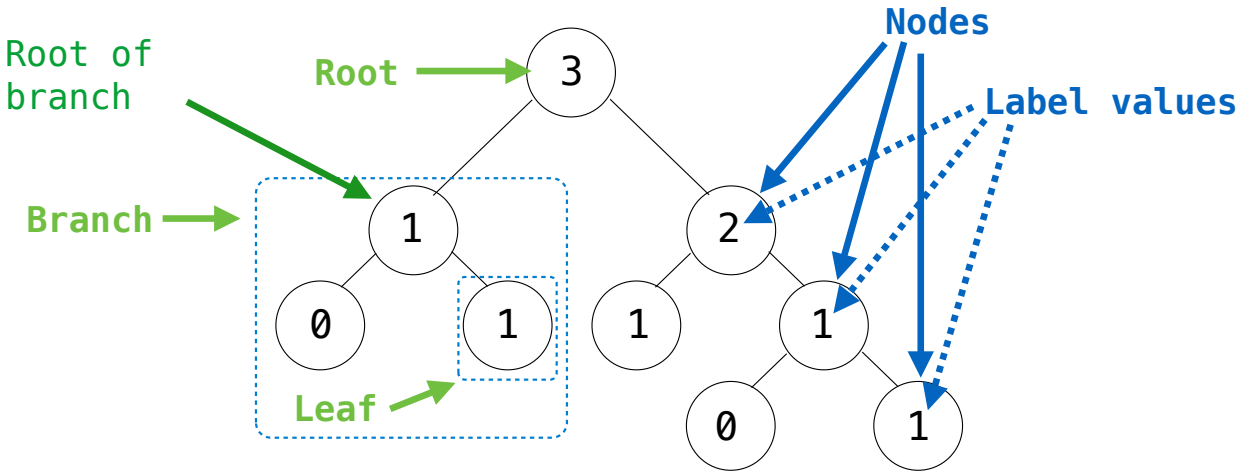

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Relative description (family trees):

Each location in a tree is called a **node**

Each **node** has a **label value**

One node can be the **parent/child** of another

People often refer to values by their locations: "*each parent is the sum of its children*"

Implementing the Tree Abstraction

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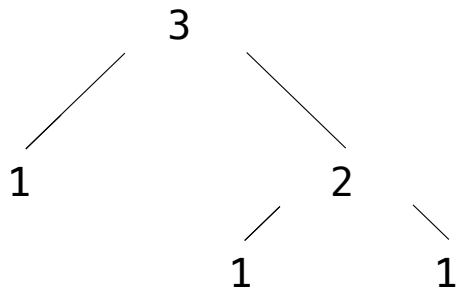

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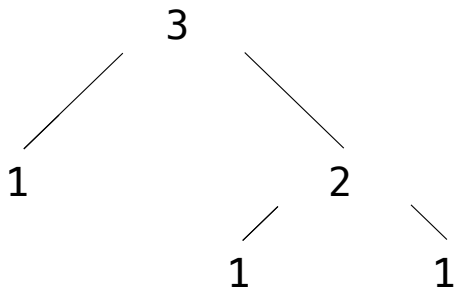


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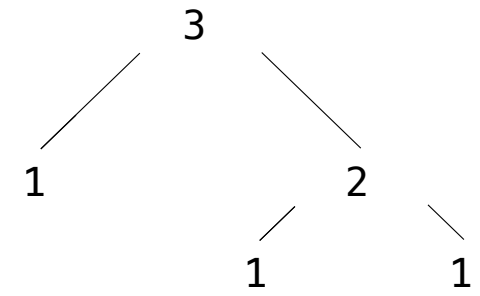


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def tree(label, branches=[]):
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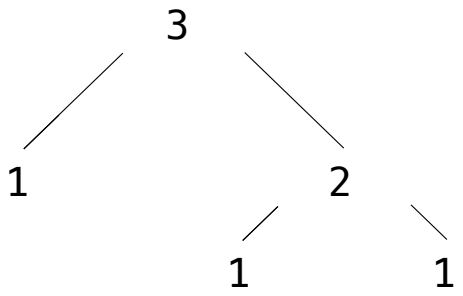


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def tree(label, branches=[]):
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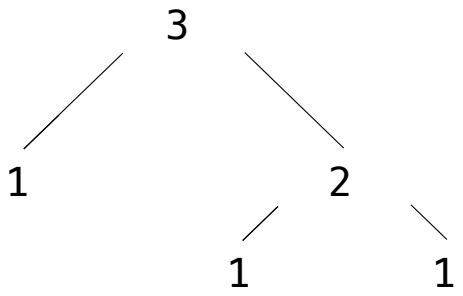


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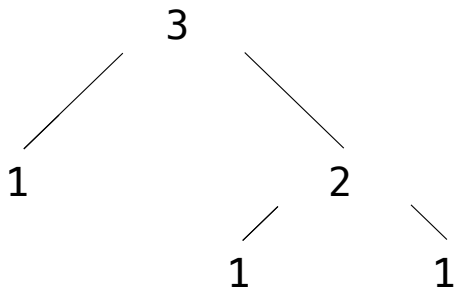


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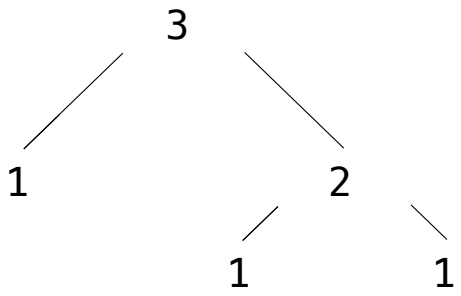


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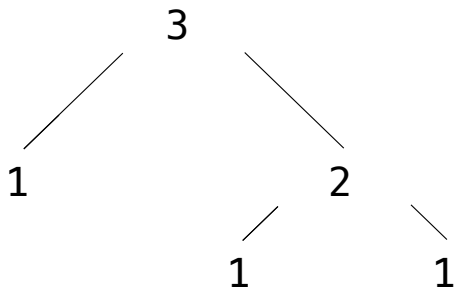


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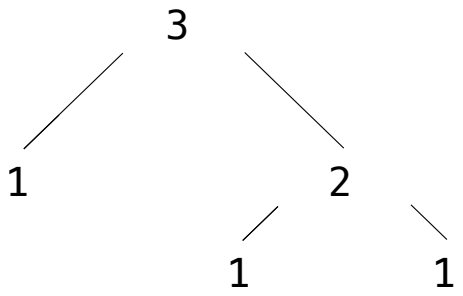


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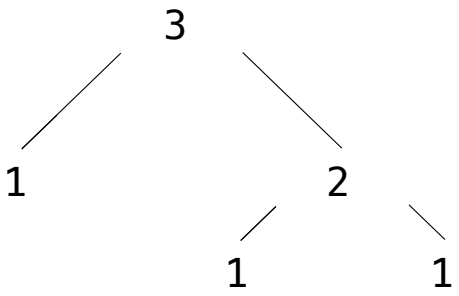


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Creates a list from a sequence of branches

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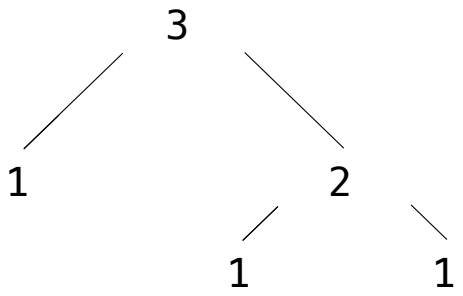


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Verifies the tree definition

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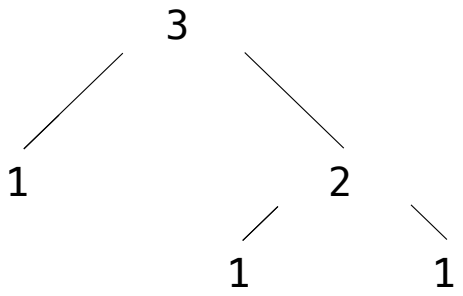


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def is_tree(tree):
    if type(tree) != list or len(tree) < 1:
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    for branch in branches(tree):
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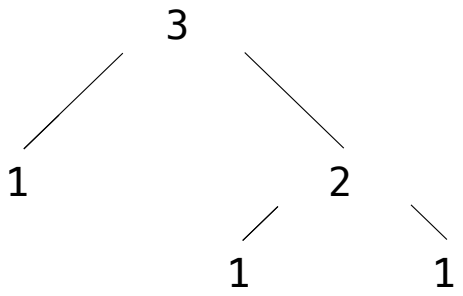


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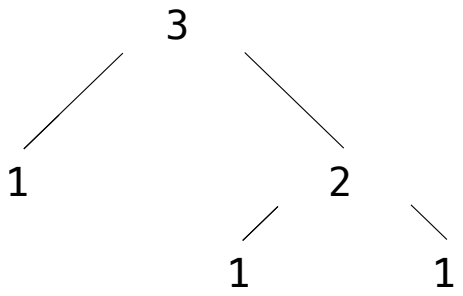
Creates a list from a sequence of branches

Verifies that tree is bound to a list

- A tree has a label value and a list of branches

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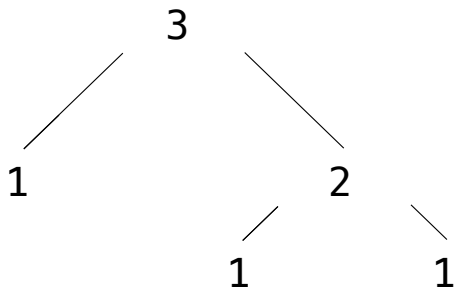


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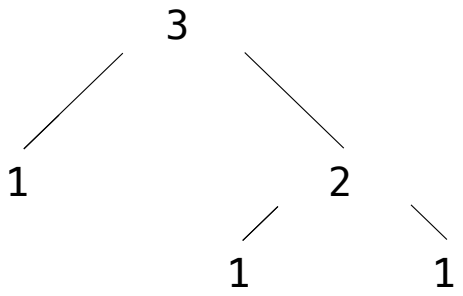


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(Demo)

Tree Processing

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(Demo)

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        return 1
    else:
        branch_counts = [count_leaves(b) for b in branches(t)]
        return sum(branch_counts)
```

(Demo)

Discussion Question

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Implement `leaves`, which returns a list of the leaf labels of a tree

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def leaves(tree):
    """Return a list containing the leaves of tree.

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    [1, 0, 1, 0, 1, 1, 0, 1]
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    >>> leaves(fib_tree(5))
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    if is_leaf(tree):
        return [label(tree)]
    else:
        return sum(_____________________________________, [])
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```
branches(tree)                          [b for b in branches(tree)]
leaves(tree)                            [s for s in leaves(tree)]
[branches(b) for b in branches(tree)]   [branches(s) for s in leaves(tree)]
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    """
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        return [label(tree)]
    else:
        return sum(List of leaves for each branch, [])
```

```
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```

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def increment_leaves(t):
    """Return a tree like t but with leaf values incremented."""
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    else:
        bs = [increment_leaves(b) for b in branches(t)]
        return tree(label(t), bs)
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```

Example: Printing Trees

(Demo)